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Maximizing Charging Efficiency of Lithium-Ion and Lead-Acid Batteries Using Optimal Control Theory

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***Abstract*— Optimal charging of stand-alone lead-acid and lithium-ion batteries is studied in this paper. The objective is to maximize the charging efficiency. In the lithium-ion case two scenarios are studied. First only electronic resistance is considered and in the next step the effect of polarization resistance is also included. By considering constant model parameters for the lithium-ion battery analytical solutions exists for both scenarios using Pontryagins minimum principle. In lead-acid chemistry the variation of total internal resistance with state of charge (SOC) is considerable and the optimal charging problem results in a set of two nonlinear differential equations with one initial and a final condition to be satisfied. This so called two point boundary value problem is solved numerically.**

I. INTRODUCTION

Batteries have become an indispensable part of our daily life. They can be found everywhere, powering our electronic gadgets, computers, and phones; they have made possible electrifying the transportation sector, hybrid and electric cars, and form a critical part of modern energy grids with renewable energy sources. The need for increased energy and power density and longer cycle life has spurred much research and development towards more efficient batteries and has also called for more effective battery management systems that monitor and control cell voltages and temperatures. Despite the recent developments, the limitations in power density and also performance reduction at low (high) temperatures still exist due to high internal resistance of batteries. A major performance bottleneck in stand-alone or hybridized battery systems is due to resistive losses during charge and discharge cycles. When hybridized (e.g. using supercapacitors [1]), there are extra degree(s) of freedom for shaping the battery charge and discharge profile and the energy management strategy can be designed with the goal of reducing resistive losses and increasing overall system efficiency.

In stand-alone operation and during discharge, the cycle is often imposed by the required load and therefore there is little that can be done in reducing resistive losses. During charging however, there is the opportunity to choose the charging time and profile such that resistive losses are reduced. Battery manufacturers often have a recommended charging profile which may be sub-optimal.

Optimal charging of lithium-ion batteries is studied in [2] which has focused on minimizing the charging time while satisfying specific physical and thermal constraints. In [3] the focus has been on optimal charging of stand-alone supercapacitors and during regenerative braking by minimizing ohmic losses. Suthar *et al.* in [4] use a single-particle model and aim to find the optimal current profile with the objective of maximizing the charge stored in the cell in a given time and constraints of minimal damage to the electrode particles during intercalation. Bashash *et al.* in [5] focus on optimizing the timing and rate of charging of a plug-in hybrid electric vehicle from the power grid where the goal is to simultaneously minimize the total cost of fuel and electricity and the total battery health degradation. Optimizing the battery charging power in photovoltaic battery systems is studied in [6] where different objectives such as charging time, battery life time, and cost of charging are considered.

The equivalent electric circuit depicted in Fig. 1 is used to represent the electrical model of the battery in this paper. In this figure OCV and V_T represent the open circuit voltage and terminal voltage of the battery, respectively.

In this model R_s indicates the ionic and electronic resistance of electrolyte and also the electronic resistance of the electrode. Charge-transfer resistance R_1 is in parallel with the double layer capacitance C_1 which is formed at the interface between the electrode and electrolyte [7]. At steady state the sum of R_s and R_1 is called the total internal resistance (R) in this paper. In this study the objective is to maximize the charging efficiency by minimizing the resistive losses in a given charging time and a specified range of the battery SOC. The charging event is assumed to be conducted in a constant ambient temperature. The charging times of interest are the ones that meet a standard charging method recommended by the manufacturer. The reason is that in such charging conditions the change in temperature of the cell and also the physical stresses are minimum. This allows to assume temperature independence of the model parameters and also neglect the possibilities of thermal runaways.

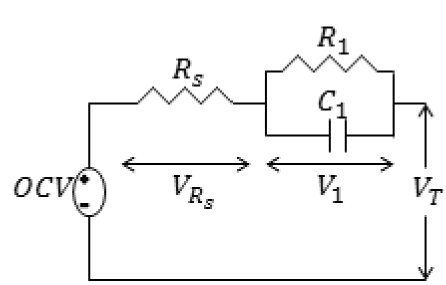

Fig. 1. Schematic of the single RC model

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The optimal charging problem for the lithium-ion battery is formulated in two steps. In the first step only R_s is considered. With the standard charging assumption, the dependence of R_s on temperature is negligible. The dependence of R_s on SOC is also shown to be negligible. In the second step the R-C branch is added to the model and the optimal control problem is solved by taking into account the effect of charge transfer resistance and double layer capacitance. The simplifying assumption of constant R_1 and C_1 results is analytical solution for the optimal charging current. This result could be used as a first step for researchers when approaching the problem by considering the dependence of model parameters on SOC and even temperature.

The optimal charging problem for the lead-acid battery is formulated similar to the first scenario in the lithium-ion battery except that the total internal resistance (R) is modeled. The efficiency maximization problem is solved by considering the dependence of the total internal resistance on SOC. This problem structure results in a two point boundary value problem with two nonlinear differential equations. Numerical methods are used to solve this problem.

II. OPTIMAL CHARGING FORMULATION OF THE BATTERY

The objective of this optimal control problem for both battery chemistries is to maximize the charging efficiency by minimizing the ohmic losses. The battery SOC is an indicator of the amount of charge stored in the battery at each time normalized by the maximum acceptable charge. The dynamics of the SOC as the common state $x_1(t)$ for both lithium-ion and lead-acid batteries, is derived by performing coulomb counting, using the current fed into the battery. Considering the charging current as the single input $u(t)$ to the system, the state equation is governed by the following differential equation:

$$\frac{d}{dt}x_1(t) = \frac{u(t)}{q_{max}} \tag{1}$$

where q_{max} is the nominal battery capacity. The optimal charging problem formulation for lithium-ion and lead-acid batteries is described in the following sections.

A. *Optimal Charging of the Lithium-Ion Battery*

The lithium-ion battery used in this study represents the *LiFePO*4 chemistry. The cell (ANR26650) has a nominal voltage and capacity of 3.3 V and 2.5 Ah, respectively [8].
1) First scenario: In this scenario only R_s as depicted in Fig. 1 is considered. The value of R_s is constant and equal to 0.01Ω. This value is obtained at 25°C by parameterizing the equivalent electric circuit model of the cell using pulse-relaxation tests and minimizing the least square error between the experimental and modeled terminal voltages [9].

The cost function to be minimized is the ohmic losses associated with R_s during the given charging time t_f governed by the following equation:

$$J_1 = \int_0^{t_f} R_s u(t)^2 dt \tag{2}$$

Following the variational approach in optimal control and utilizing the Pontryagin's minimum principle the Hamiltonian is formed as follows:

$$H(x,u,t) = R_s u(t)^2 + \lambda_1(t)\frac{u(t)}{q_{max}} \tag{3}$$

where $\lambda_1(t)$ is the Lagrange multiplier or the co-state. The necessary conditions of optimality should be satisfied as follows:

$$-\frac{\partial H}{\partial x_1} = \frac{d\lambda_1(t)}{dt}, \qquad \frac{\partial H}{\partial u} = 0 \tag{4}$$

Knowing that q_{max} and R_s are constant parameters, the Hamiltonian in equation (3) will only be a function of the system input and therefore $\frac{\partial H}{\partial x_1}$ will be zero in equation (4). This indicates that the derivative of the co-state with respect to time is zero and the co-state should be a constant. Taking the derivative of the Hamiltonian with respect to the input and setting it to zero according to equation (4), results in the following optimal charging current:

$$u^*(t) = -\frac{1}{2}\frac{1}{R_s q_{max}}\lambda_1 \tag{5}$$

where * denotes the optimal solution. The parameters λ_1, R_s and q_{max} are all constants in equation (5) which indicates that the resulting optimal charging current is also constant. Integrating equation (1) with the knowledge of $u(t)$ being constant and using the boundary conditions $x_1(0) = SOC_i$ and $x_1(t_f) = SOC_f$, the value of this optimal and constant charging current is derived as follows:

$$u^*(t) = \frac{q_{max}(SOC_f - SOC_i)}{t_f} \tag{6}$$

This is in fact the minimizing solution since $\frac{\partial^2 H}{\partial u^2} = 2R_s > 0$. Given a specific charging time, the most efficient way to charge the battery will be applying a constant current equal to equation (6). For example the optimal strategy of charging the battery from zero charge to full charge in one hour is to apply a constant current equal to 2.5 A. Smaller constant currents compared to the optimal constant current result in lower resistive losses but will not meet the required charging time. On the other hand higher constant currents compared to the optimal constant current, result in faster charging but with higher resistive losses. The standard charging method recommended by the manufacturer is to charge the battery with a constant current-constant voltage (CC-CV) protocol at a rate of 1C (2.5A).
2) Second scenario: In this scenario the R-C branch is added to the model to include the effect of the polarization resistance R_1. The value of R_1 is assumed to be constant and not a function of temperature or SOC. The value for R_1 is 0.016Ω which is an average value over the SOC range [9].

Assume I_1 and I_2 are the currents passing through R_1 and C_1, respectively. Applying the Kirchoff's current and voltage laws to the R-C branch the second state equation governing the dynamics of the current passing through R_1 is obtained.

The problem in this case is to obtain the optimal charging current for a second order system governed by the following state equations:

$$\frac{d}{dt}x_1(t) = \frac{u(t)}{q_{max}}, \qquad \frac{d}{dt}x_2(t) = \frac{1}{R_1C_1}[u(t) - x_2(t)] \qquad (7)$$

where the two states x_1 and x_2 are the SOC of the battery and the current passing through the polarization resistance R_1. The objective, similar to the first scenario, is to maximize the charging efficiency. The difference is that the contribution of the polarization resistance to the total ohmic losses is also considered. The cost function to be minimized is:

$$J_2 = \int_0^{t_f} [R_s u(t)^2 + R_1 x_2(t)^2] dt \qquad (8)$$

The Hamiltonian in this case is given by the following equation:

$$H(x,u,t) = R_s u(t)^2 + R_1 x_2(t)^2 + \lambda_1(t)\frac{u(t)}{q_{max}} + \frac{\lambda_2(t)}{R_1C_1}[u(t) - x_2(t)] \qquad (9)$$

The necessary conditions for optimality are:

$$-\frac{\partial H}{\partial x_1} = \frac{d\lambda_1(t)}{dt}, \quad -\frac{\partial H}{\partial x_2} = \frac{d\lambda_2(t)}{dt}, \quad \frac{\partial H}{\partial u} = 0 \qquad (10)$$

From the first two conditions the dynamics of the co-states are derived and from the third condition the optimal input is obtained as follows:

$$u^*(t) = (\frac{-1}{2R_s q_{max}})\lambda_1(t) + (\frac{-1}{2R_s R_1 C_1})\lambda_2(t) \qquad (11)$$

Substituting the optimal input into the state equations in (7), the optimal state dynamics are derived. The result is the following set of four linear first order ordinary differential equations (ODE):

$$\begin{aligned} \frac{d}{dt}x_1(t) &= a_1\lambda_1(t) + a_2\lambda_2(t) \\ \frac{d}{dt}x_2(t) &= b_1 x_2(t) + b_2\lambda_1(t) + b_3\lambda_2(t) \\ \frac{d}{dt}\lambda_1(t) &= 0 \\ \frac{d}{dt}\lambda_2(t) &= c_1 x_2(t) + c_2\lambda_2(t) \end{aligned} \qquad (12)$$

where a_1, a_2, b_1, b_2, b_3, c_1 and c_2 are constant parameters equal to:

$$\begin{aligned} a_1 &= \frac{-1}{2R_s q_{max}^2}, \quad a_2 = \frac{-1}{2R_s R_1 C_1 q_{max}}, \\ b_1 &= \frac{-1}{R_1C_1}, \quad b_2 = \frac{-1}{2R_s R_1 C_1 q_{max}}, \\ b_3 &= \frac{-1}{2R_s R_1^2 C_1^2}, \quad c_1 = -2R_1, \quad c_2 = \frac{1}{R_1C_1} \end{aligned} \qquad (13)$$

Solving this system of linear ODE's simultaneously results in four algebraic equations with four unknowns. The unknown constants are obtained by applying the boundary conditions specific to this problem which consist of two initial and two final conditions. The initial and final condition for $x_1 = SOC$ are:

$$x_1(t_0) = SOC_i, \quad x_1(t_f) = SOC_f, \qquad (14)$$

where SOC_i and SOC_f are specified according to the desired range of charging. In this specific problem the charging time is specified and fixed while the values of the second state at the initial and final time are free; this results in the following equations for the remaining two boundary conditions [10]:

$$\begin{aligned} \frac{\partial h}{\partial x_2}(x_2(t_0)) &= \lambda_2(t_0) = 0 \rightarrow \text{Initial condition for } x_2 \\ \frac{\partial h}{\partial x_2}(x_2(t_f)) &= \lambda_2(t_f) = 0 \rightarrow \text{Final condition for } x_2 \end{aligned} \qquad (15)$$

In general $h(x(t_f), t_f)$ is the term involving the final states and final time in the cost function which in this study is zero. Given all boundary conditions, one can solve for the states, co-states, and the optimal input. Consider charging a battery cell from zero charge SOC_i = $x_1(0)$ = 0 to full charge SOC_f = $x_1(t_f)$ = 1 in one hour. The results for this example are depicted in Fig. 2.

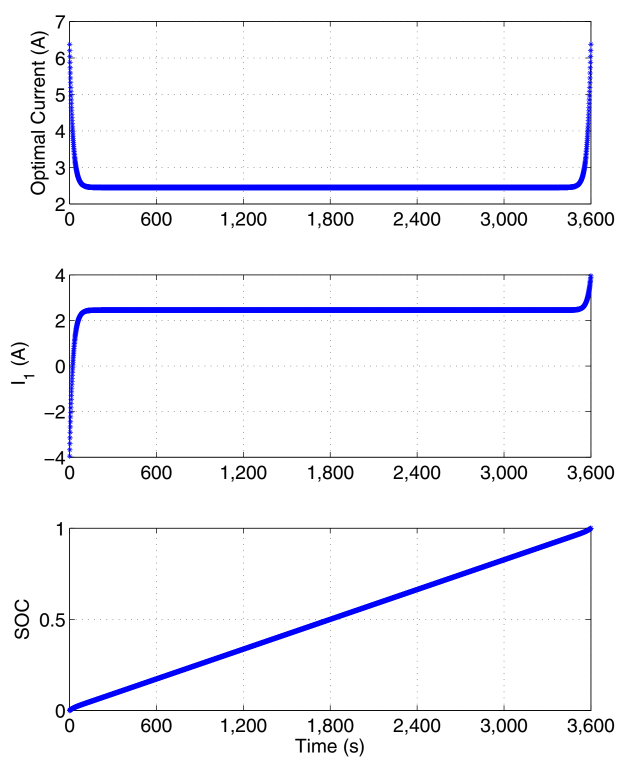

Fig. 2. Optimal charging current, I_1, and SOC for charging the lithium-ion battery from zero to full charge in 1 hour

The optimal charging current for this scenario is slightly different from the result of the first scenario. The optimal input in this case is almost a constant current equal to 2.5 A in the majority of times. It may be insightful to also show the result for a fast charging case. Fig. 3 shows the optimal charging current and the two states of the system when the cell is charged from zero to full charge in 5 minutes.

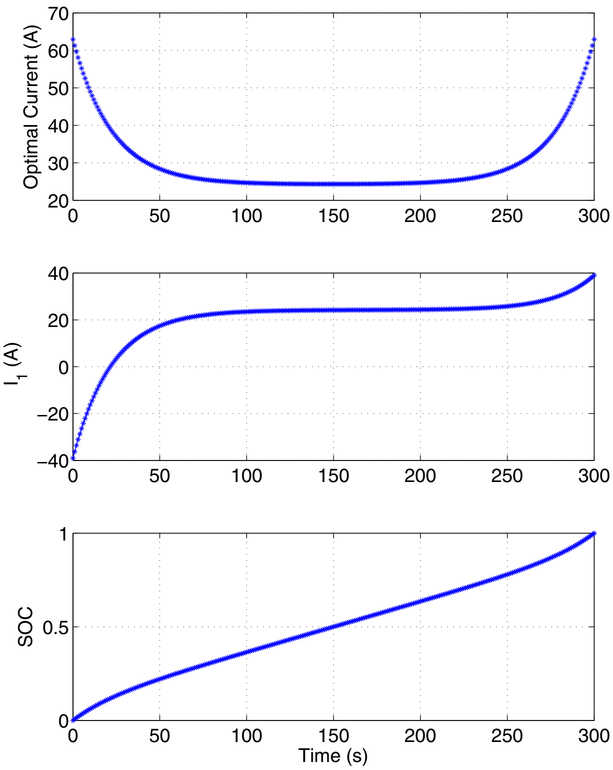

Fig. 3. Optimal charging current, I_1, and SOC for charging the lithium-ion battery from zero to full charge in 5 minutes

This charging strategy is not practical due to thermal and physical constraints plus safety and battery degradation problems. Although it may be interesting to observe that by reducing the charging time the optimal profile differs from the constant current result observed in the first scenario and also slow charging in the second scenario.

B. Lithium-Ion Efficiency Analysis

The open circuit voltage (OCV) of the lithium-ion battery used in this study as a function of SOC is depicted in Fig. 4 [9]. As shown in the figure, the OCV can be approximated by a linear function using the OCV data in the SOC range of 10 to 95 percent. This linear function fitting and also the assumption of a constant total internal resistance (R = R_s + R_1 = 0.026Ω) makes the analytical efficiency analysis possible. The linear approximation of the OCV is governed by:

$$V(t) = aSOC(t) + b \tag{16}$$

where $V(t)$ is the open circuit voltage of the battery.

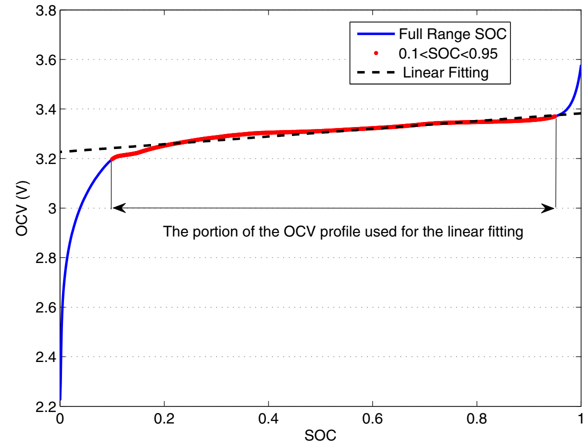

Fig. 4. OCV versus SOC for the lithium-ion battery

In order to find the optimal charging efficiency, the total energy stored in the battery and energy loss is required. The efficiency is then:

$$\rho = \frac{E_{Battery}}{E_{Battery} + E_{Loss}} \tag{17}$$

The energy loss in the battery during optimal charging is already known and is equal to $\int_0^{t_f} Ru_1^*(t)^2 dt$ where R is the total internal resistance. The total energy stored in the battery is:

$$\begin{aligned} E_{Battery} &= \int_0^{t_f} V(t)I(t)dt \\ &= \int_0^{t_f} V(t)\frac{dq(t)}{dt}dt \\ &= \int_{q_i}^{q_f} V(t)dq \end{aligned} \tag{18}$$

where $I(t)$, $V(t)$ and $q(t)$ are the battery current, OCV and the charge in ampere-hours, respectively. The relationship between OCV and the charge stored in the battery is obtained by considering the fact that:

$$SOC(t) = \frac{q(t)}{q_{max}} \tag{19}$$

where q_{max} is the nominal capacity of the battery in ampere-hours (Ah). Substituting for SOC in equation (16) from equation (19) the linear relationship between $V(t)$ and $q(t)$ is obtained as follows:

$$V(t) = \frac{a}{q_{max}}q(t) + b \tag{20}$$

where a and b for this specific battery are 0.156 and 3.226, respectively. The nominal capacity of the battery is 2.5 Ah. Performing the integration by substituting $V(t)$ from equation (20) into equation (18) and using the SOC definition in equation (19) for the initial and final SOC, the maximum energy stored in the battery is obtained as follows:

$$E_{Battery} = q_{max}(SOC_f - SOC_i)[\frac{a}{2}(SOC_f + SOC_i) + b)] \quad (21)$$

where the unit for energy is watt-hours (Wh). The real maximum amount of energy which the battery can store is obtained from integrating the original $OCV - q$ profile which results in 8.2 Wh for the cell used in this study. Using equation (21) and charging the battery from zero charge to full charge, the maximum energy stored in the battery is calculated as 8.26 Wh. This illustrates that the linear approximation of OCV for lithium ion battery is an effective approach to perform analytical efficiency analysis. Substituting the expressions for E_{loss} and $E_{battery}$ in equation (17), the optimal charging efficiency of lithium-ion battery is obtained as follows:

$$\rho^* = \frac{1}{1 + \frac{Rq_{max}(SOC_f - SOC_i)}{t_f(\frac{1}{2}a(SOC_f + SOC_i) + b)}} \quad (22)$$

where t_f is the charging time in hours. Consider charging the lithium-ion battery from some initial charge SOC_i to full charge ($SOC_f = 1$) then Fig. 5 shows the optimal efficiency as a function of t_f/Rq_{max} and for four different initial SOCs. The result shows that starting the charging from a higher initial SOC results in better charging efficiency.

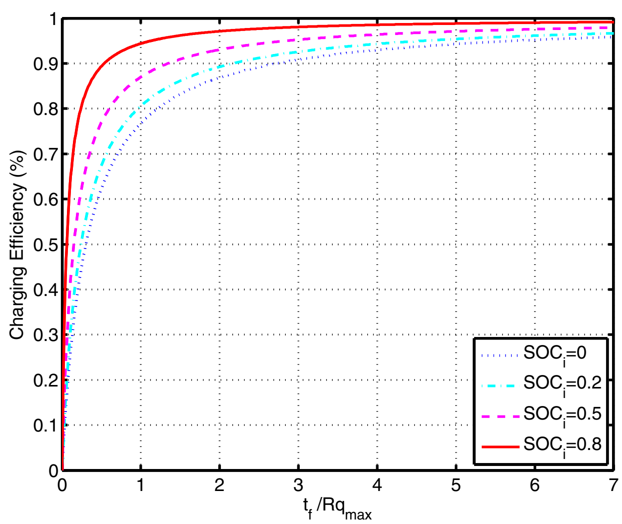

Fig. 5. Optimal efficiency versus t_f/Rq_{max} for different initial SOC in lithium-ion battery

C. Optimal Charging of the Lead-Acid Battery

The lead-acid battery used in this study is a AP-12220EV-NB module with nominal voltage and capacity of 12 volts and 22 Ah, respectively [11]. The real capacity of the module is 19.7 Ah which is obtained by discharging the fully charged module with a small current of 0.55A, from the upper to the lower voltage limit. Similar to the lithium-ion battery, specifically designed pulse-relaxation tests such as the method used in [12] is utilized to derive the total internal resistance of the cell (R) as a function of SOC. Fig. 6 shows that the total internal resistance for a lead-acid battery is strongly dependent on SOC.

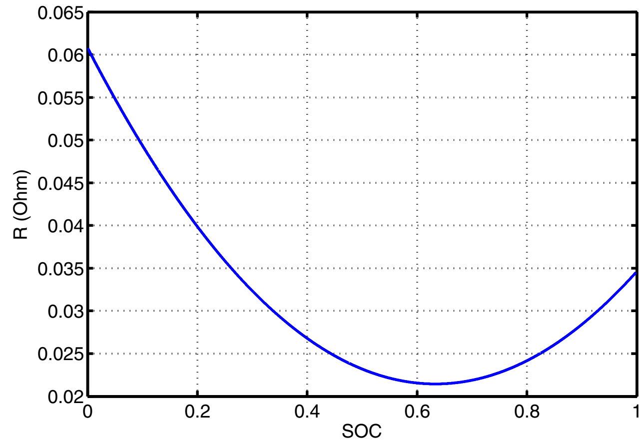

Fig. 6. Total internal resistance versus SOC for the lead-acid battery

The lead-acid battery is modeled by a single internal resistance and the only state is the state of charge of the battery governed by equation (1). The optimal control is subject to minimize the losses associated with the total internal resistance. Therefore the Hamiltonian is:

$$H(x,u,t) = R(x_1)u(t)^2 + \lambda_3(t)\frac{u(t)}{q_{max}} \quad (23)$$

where R is the total internal resistance and $\lambda_3(t)$ is the co-state. Here $R(x_1)$ is approximated by a second order polynomial function of the state x_1 as follows:

$$R(x_1) = 0.098x_1^2 - 0.12x_1 + 0.061 \quad (24)$$

The necessary conditions to be satisfied are:

$$-\frac{\partial H}{\partial x_1} = -\frac{dR(x_1)}{dx_1}u^2(t) = \frac{d\lambda_3(t)}{dt} \quad (25)$$

$$\frac{\partial H}{\partial u} = 2R(x_1)u(t) + \frac{\lambda_3(t)}{q_{max}} = 0 \quad (26)$$

Solving for $u(t)$ in equation (26) the optimal charging current is obtained as follows:

$$u^*(t) = -\frac{1}{2q_{max}}\frac{1}{R(x_1)}\lambda_3(t) \quad (27)$$

Substituting $u^*(t)$ from equation (27) in equations (1) and (25), the following set of two coupled nonlinear ODEs are obtained:

$$\frac{dx_1(t)}{dt} = -\frac{1}{2q_{max}^2}\frac{1}{R(x_1)}\lambda_3(t) \quad (28)$$

$$\frac{d\lambda_3(t)}{dt} = -\frac{1}{4q_{max}^2}\frac{dR(x_1)}{dx_1}\frac{1}{R^2(x_1)}\lambda_3^2(t) \quad (29)$$

Charging the lead-acid battery in t_f units of time from zero to full charge requires the following initial and final conditions to be satisfied:

$$x(0) = SOC_i, \qquad x(t_f) = SOC_f \tag{30}$$

The system of two nonlinear ODEs with one initial and another final condition forms a two point boundary value problem which could only be solved using numerical methods. The optimal charging current is obtained by solving for $\lambda_3(t)$ and $x_1(t) = SOC(t)$ and substituting in equation (27).

III. Numerical Results

In this section the numerical solution for the optimal charging of the lead-acid battery is presented.

Optimal charging problem for lead-acid battery was formulated in the previous section. The result was a set of two nonlinear ODEs with one initial and another final condition. One way to solve this system of ODEs is to specify the initial condition for the SOC and iteratively guess the initial condition for λ_3 until SOC reaches the final specified value. This method could be applied by using *ODE* solvers in Matlab. Consider the case of charging the lead-acid battery module from zero to full charge in one hour. Fig. 7 shows the variation of optimal charging current, SOC, and λ_3 by time.

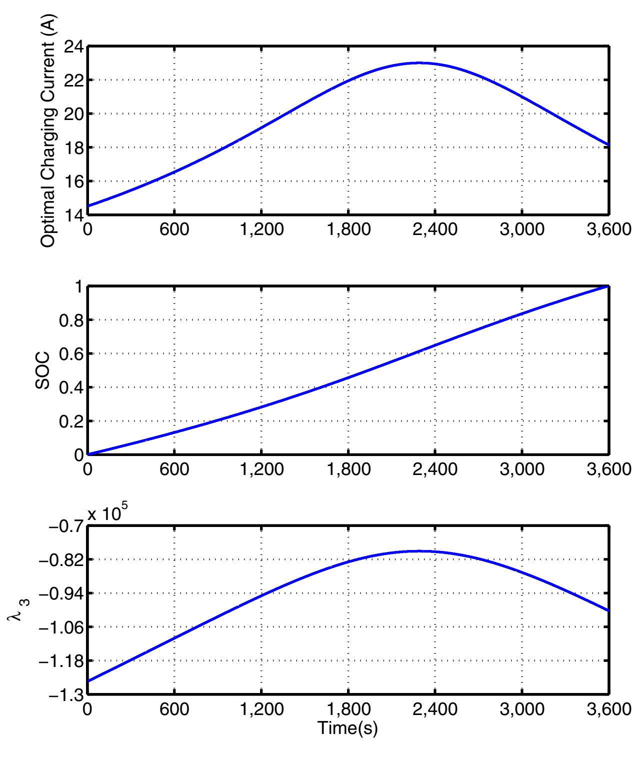

Fig. 7. Optimal charging current, SOC, and λ_3 profiles for charging the lead-acid battery from zero to full charge in 1 hour

As shown in the numerical results the optimal charging current for lead-acid battery is not a constant current profile similar to the first scenario in lithium-ion batteries. In order to compare the constant current charging of the lead-acid battery with the optimal charging strategy, the energy losses in both methods are calculated. For the case of charging the lead-acid battery from zero to full charge in one hour the energy losses due to the resistive losses with the optimal charging strategy are 46.18 KJ compared to 48.9 KJ for constant current charging. This is a 5.5% of less energy converted to heat which could be significant in terms of thermal management of battery packs.

IV. Conclusion

This paper investigated the optimal charging strategy for lithium-ion and lead-acid batteries. Formulating the optimal control problem and utilizing Pontryagin's minimum principle, analytical result existed for the lithium-ion battery under certain assumptions. In the lead-acid battery case the results show that the optimal charging current is not necessarily constant. Constant current charging of lead-acid battery results in 5.5% higher thermal heating which could be considered in thermal management studies of lead-acid batteries. The simplifying assumptions made in this study, sets the ground for studies on the battery optimal charging problem in the future. One direction that we will pursue is applying appropriate thermal constraints to meet the challenges in problems such as fast charging where the temperature variation and its effect on model parameters plays a significant role.

References

[1] Y. Parvini, J. B. Siegel, A. G. Stefanopoulou, and A. Vahidi, "Preliminary results on identification of an electro-thermal model for low temperature and high power operation of cylindrical double layer ultracapacitors," in *American Control Conference (ACC), 2014*. IEEE, 2014, pp. 242–247.

[2] R. Klein, N. Chaturvedi, J. Christensen, J. Ahmed, R. Findeisen, and A. Kojic, "Optimal charging strategies in lithium-ion battery," in *American Control Conference (ACC), 2011*, June 2011, pp. 382–387.

[3] Y. Parvini and A. Vahidi, "Optimal charging of ultracapacitors during regenerative braking," in *Electric Vehicle Conference (IEVC), 2012 IEEE International*. IEEE, 2012, pp. 1–6.

[4] B. Suthar, V. Ramadesigan, S. De, R. D. Braatz, and V. R. Subramanian, "Optimal charging profiles for mechanically constrained lithium-ion batteries," *Physical Chemistry Chemical Physics*, vol. 16, no. 1, pp. 277–287, 2014.

[5] S. Bashash, S. J. Moura, J. C. Forman, and H. K. Fathy, "Plug-in hybrid electric vehicle charge pattern optimization for energy cost and battery longevity," *Journal of Power Sources*, vol. 196, no. 1, pp. 541–549, 2011.

[6] J. Li and M. A. Danzer, "Optimal charge control strategies for stationary photovoltaic battery systems," *Journal of Power Sources*, vol. 258, pp. 365–373, 2014.

[7] S. Sato and A. Kawamura, "A new estimation method of state of charge using terminal voltage and internal resistance for lead acid battery," in *Power Conversion Conference, 2002. PCC-Osaka 2002. Proceedings of the*, vol. 2. IEEE, 2002, pp. 565–570.

[8] *Nanophosphate High Power LithiumIon Cell ANR26650M1-B*, A123 Systems.

[9] X. Lin, H. E. Perez, S. Mohan, J. B. Siegel, A. G. Stefanopoulou, Y. Ding, and M. P. Castanier, "A lumped-parameter electro-thermal model for cylindrical batteries," *Journal of Power Sources*, vol. 257, pp. 1–11, 2014.

[10] D. Kirk, *Optimal Control Theory: An Introduction*. Mineola, New York: Dover Publishers, 2004.

[11] *AP-12220EV Technical Specifications*, Amstron Power Solutions.

[12] S. Fiorenti, J. Guanetti, Y. Guezennec, and S. Onori, "Modeling and experimental validation of a hybridized energy storage system for automotive applications," *Journal of Power Sources*, vol. 241, pp. 112–120, 2013.